**Lesson 0**

C4: General vector spaces

dimensional Euclidean Space

: One-dimensional space

Can be represented as a straight line

Corresponds with the set of real numbers

Written as the ordered tuple

: Two-dimensional space

Can be represented geometrically as a plane

Corresponds with two mutually perpendicular copies of , called the axis and axis

Origin denoted , is the point

Written as the ordered pair

: Three-dimensional space

Can be represented geometrically as a plane

Corresponds with three mutually perpendicular copies of , called the axis, axis and axis

Origin denoted , is the point

Written as the ordered tuple

A drawing of a house

Description automatically generated with medium confidence *right hand rule coordinate planes in*

A picture containing ski tow, lamp, linedrawing

Description automatically generated

**Lesson 1**

C4: Real vector spaces

* These are defined by axioms (assumptions)
* There are different types of vector spaces: Real, Polynomial, subspaces,

**Operations on Real vector spaces**

|  |  |
| --- | --- |
| Addition | Given two elements x, y in X, one can form the sum x+y, which is also an element of X. |
| Inverse | Given an element x in X, one can form the inverse -x, which is also an element of X. |
| Scalar multiplication | Given an element x in X and a real number c, one can form the product cx, which is also an element of X |

Axioms on Vector Spaces

* Ten axioms (assumptions)
* 8 of these are properties of vectors in

**Additive Axioms**

1.

*If and are objects in ,*

*Then is in*

2.

3.

4. *There is an object in , called a zero vector for*

*such that*

5. *For each in*

*There is an object in , called a negative of*

*such that*

**Distributive Axioms**

6.

*If is any scalar and is any object in*

*then is in*

7.

8.

**Multiplicative Axioms**

9.

10.

Example: Is a vector space?

Yes.

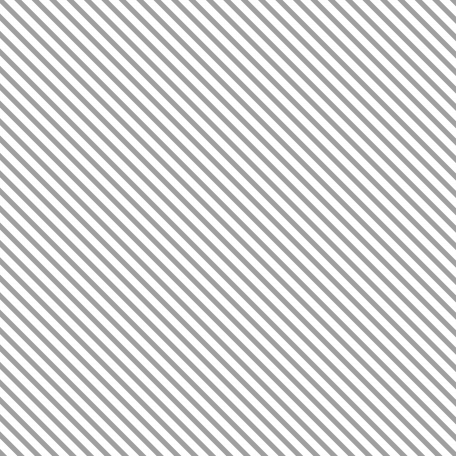
Example: Is a vector space?

No. Via Axiom 6 the vector would be outside

Chart, line chart

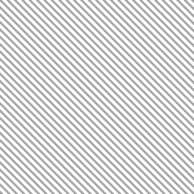
Description automatically generatedChart, table

Description automatically generated



**✘**

**✓**



6.

*If is any scalar and is any object in then is in*

e.g. let

1.

*If and are objects in ,*

*Then is in*

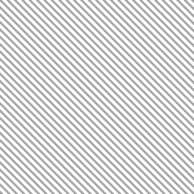
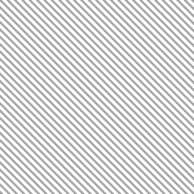
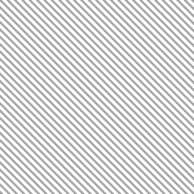
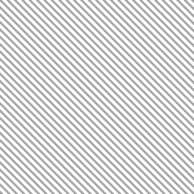
Example: Is a vector space?

No. Via Axiom 6 the vector would be outside

Chart, line chart

Description automatically generatedChart, line chart

Description automatically generated



**✘**

**✓**

6.

*If is any scalar and is any object in then is in*

1.

*If and are objects in ,*

*Then is in*

**Lesson 2**

C4: Real vector spaces

Polynomials

is a vector space

* The degree of a polynomial is the highest power with a non-zero coefficient

*Assuming*

* The polynomial with degree is the zero polynomial

Axioms on Polynomial Vector Spaces

*Assuming*

1.

*Two vector spaces are in independently.*

*Their sum is also in*

e.g.

Rewritten:

2.

3. for all

**Lesson 3**

C4: Real vector spaces

Subspaces

- Subspaces are also vector spaces

*is a subspace of*

A subspace of a vector is a subset that satisfies three conditions:

1. The zero vector is in

2. is closed under addition

*A set is closed under addition if you can add any two numbers in the set and still have a number in the set as a result*

then

3. is closed under multiplication of scalars.

defined on

then

Some notable examples:

A subspace is also a vector space **✓**

A vector space is also a subspace **✓**

is a subspace of ✘

✘ Not a

Subspace!

Remember every vector in a subspace must come from the vector space

**✓** is a

Subspace!

<https://www.youtube.com/watch?v=TuOQ4-kmcE0>

Example: Is a subspace of

Yes.

1. **✓**
2. **✓**
3. **✓**

Example: Let .

Show that is a subspace of

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

1. **✓**
2. Define and

**✓**

*This is the span*

1. **✓**

Remember that we defined

*Also in the span*

Example: Let be the set of all vectors of the form

Show that is a subspace or find a counterexample.

Reduce this to a span of something

**✓**

**Lesson 4**

C4: Real vector spaces

Span of vectors

* Kind of like the cross product of

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

With weights

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors

With weights

Example:

Let

*Write down all the possible combinations*

Chart, line chart

Description automatically generated

Background pattern

Description automatically generated Unit vectors.

So the x axis is the plane

And the y axis is the plane

*Therefore, the span of our vectors and is the whole plane*

AKA: *the vectors and span the whole of*

Example:

List 3 vectors in where and

OR

OR

OR

OR

etc.

*Just remember to list ALL possible* linear combinations

Example:

Given

*matrix vector*

Is in

*B would be a linear combination of and*

1. Make an augmented matrix with and

1. Solve augmented matrix

*Add 2x row one to row 3*

*Add 2x row one to row 3, multiply row by -1*

Solutions

R3:

R2:

R1:

**Lesson 5**

C4: Real vector spaces

Linear combinations

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

With weights

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors

With weights

Example:

is a linear combination

AKA

Some notable examples:

Another notation for vector is ✘

*The vector*  *is 1 dimensional*

*The vector 2 dimensional (2 vectors)*

A linear combination of and is **✓**

*Remember that a linear combination is just a scalar a vector*

*Would be the same as*

is a vector in ✘

*Corresponds to the points so should be represented in a 3-dimensional space*

Example

Let

*is just a linear combination of and*

*is also a linear combination of and . We would probably assign a value to represent . Therefore, let*

Chart, line chart

Description automatically generatedBackground pattern

Description automatically generated

Example

Let

Can be generated as a linear combination of and ?

1. Make an augmented matrix with and ?

**Wolframalpha**

row reduce [(-1,-2,-7), (2,-5,-4), (-5,-6,-3)]

1. Solve for augmented matrix

Solutions:

R2:

R1:

1. Substitute solutions into and

**✓**

A vector equation

As the same solution set as the linear system whose augmented matrix is

[]

**Lesson 6**

C4: Real vector spaces

Linear independence

<https://www.youtube.com/watch?v=XI2kYIxhe-o>

A set of vectors in is linearly independent if

Has the only trivial solution

*The homogenous system (i.e. of the form ) has only the trivial solution*

If a non-trivial solution exists, then the set of vectors is linearly dependent

*If there is a free variable, then the vectors are linearly independent*

We don’t want any redundant vectors!

e.g.

Some notable examples:

If contains , then the set is linearly independent

A set of two vectors is linearly independent when

*Vector v1 is NOT equal to some multiple of v2*

**✘**

Example:

Let

**✓**

Example:

Let

is linearly dependent if

*If a set has more vectors than there are entries in each vector, then the set is linearly dependent*.

Example:

**✘**

Let

2 dimensions, 3 vectors.

Linearly dependent

Example:

Let

Chart, line chart

Description automatically generated

**✘**

The set is linearly dependent

*We can get to the point using and*

*is redundant, it to get to any new points*

Example:

Let

Chart, line chart

Description automatically generated

**✓**

The set is linearly independent

Example:

Are the following sets linearly independent?

Let

1. Make an augmented matrix with , and

1. Solve for augmented matrix

swap R1 and R2

Solutions:

R3:

R2:

R1:

Trivial solution

No free variable. Therefore, linearly independent

**Lesson 7**

C4: Real vector spaces

Basis and Dimension

<https://www.youtube.com/watch?v=OLqc_rt7abI>

Let be a subspace of vector space .

is a basis for if:

1. is linearly independent

*Linear combinations*

Some notable examples:

is the standard basis for

unit vectors

1. all linearly independent
2. all in

is the standard basis for

1. all linearly independent
2. all in

Example:

Determine if is a basis for

1. Make an augmented matrix with the set of vectors

1. Solve for augmented matrix

Using the above, Let

**✘**

Not linearly independent.

Because

Example:

Find a basis for the set of vectors in on the line

Rewrite the above as

1. all linearly independent

Dimension

[https://www.youtube.com/watch?v=zr\_iMt0k5TQ](https://www.youtube.com/watch?v=zr_iMt0k5TQ&list=PLDDGPdw7e6AjJacaEe9awozSaOou-NIx_&index=39)

If is spanned by a finite set,

then is finite dimensional

The dimension of is the number of vectors in a basis for

*TDLR:*

*is the vector*

*is the subspace in*

*is the basis of*

*is the number of vectors in*

Some notable examples:

*remember standard basis*

*remember standard basis*

Let be a subspace of the finite dimensional vector space .

Any linearly independent set in can be expanded to a basis for

and

**of nullspace and column space**

The dimension of is the number of free variables in

The dimension of is the number of pivot columns in

*Pivot columns: columns that contain the leading 1’s of the rows*

e.g., 1,2 and 4 are the pivot columns

Example:

2x4 matrix

2 pivots

2 free variables

Example:

Find the dimensions of the subspace spanned by

* Instinctively each vector has 3 points, so they’re probably in
* Because of this, it is 3-dimensional so
* Need to still prove because it might actually be subspace in a plane in

1. Make an augmented matrix with the set of vectors

1. Solve for augmented matrix

…

We see that

So, is the redundant vector.

From first attempt

Find the basis for the row space of (i.e. )

Also one of the subspaces of

**Lesson 8**

C4: Real vector spaces

Change of basis

**Lesson 9**

C4: Real vector spaces

Row space, column space and Null Space

<https://www.youtube.com/watch?v=JlC58uaJVsg>

**Nullspace**

The Null Space of an matrix , is the set of all solutions to



*Set of solutions in that map to*

*Null space = no of non-pivot columns*

Some notable examples:

If is , is a subspace of

Example:

Is in where

Using the definition, we just need to check that

times first column

times second column

times third column

**✓**

Finding explicit solutions to

Example:

Find the spanning set of

1. Write in reduced echelon form

**Column space**

If is , and

*Column space is basically a span*

*Remember that a span is a subspace*

Some notable examples:

If is , is a subspace of

What happens if has a solution for every in

No matter you use, you will always get

*Like a linear combination*

The span will be all of

Example:

Find a matrix such that

*We need to find a matrix a that produces the span*

Separate the vector in terms of

and

*Like a linear combination*

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors With weights

Let

The span of the above is a linear combination of the first and second column

Example:

Find so that

If is a subspace of , what is ?

*We need to find a matrix a that produces the span*

*What is the dimension that is a subspace of?*

Let

matrix

*Should be a subspace of*

**Row Space**

<https://www.youtube.com/watch?v=Q7x5So3suYI>

The set of all linear combinations of row vectors is called the row space

Row is a subspace of

We could write as

**wolframalpha**

transpose[(1,2,3,4), (0,1,2,3), (0,0,1,2)]

Example:

If matrices and are row equivalent, then

Find given

*A is equivalent to B*

*Different notation*

Rank

<https://www.youtube.com/watch?v=Q7x5So3suYI>

Some notable examples

**Rank Theorem**

Given ,

Example:

If and

What is

Example:

If the null space of a matrix is 5-dimensional, what is the rank of ?

Using Rank Theorem

Example:

If A is , what is the smallest possible dimension of ?

*Null space = no of non-pivot columns*

We could have 4 pivot columns i.e.,

**Lesson 11**

C5: Eigenvalues, Eigenvectors

Eigenvalues, Eigenvectors

**Lesson 12**

C5: Eigenvalues, Eigenvectors Diagonalization

**Lesson 13**

C6:

Inner Products

**Lesson 14**

C6:

Angle and orthogonality in inner product spaces

**Lesson 15**

C6:

Orthonormal bases, Gram-Schmidt Process

**Lesson 16**

C6:

Best approximation, least squares

**Lesson 17**

C7: Diagonalization

orthogonal matrices

**Lesson 18**

C7: Diagonalization

Orthogonal diagonalization

**Lesson 19**

C8: Linear Transformations

General Linear Transformations

**Lesson 20**

C8: Linear Transformations

Kernel and Range

**Lesson 21**

C8: Linear Transformations

Inverse linear transformations

**Lesson 22**

C8: Linear Transformations

Matrices of general linear transformations

**Lesson 23**

C8: Linear Transformations

Similarity