**Lesson 0**

C4: General vector spaces

dimensional Euclidean Space

: One-dimensional space

Can be represented as a straight line

Corresponds with the set of real numbers

Written as the ordered tuple

: Two-dimensional space

Can be represented geometrically as a plane

Corresponds with two mutually perpendicular copies of , called the axis and axis

Origin denoted , is the point

Written as the ordered pair

: Three-dimensional space

Can be represented geometrically as a plane

Corresponds with three mutually perpendicular copies of , called the axis, axis and axis

Origin denoted , is the point

Written as the ordered tuple

A drawing of a house

Description automatically generated with medium confidence *right hand rule coordinate planes in*

A picture containing ski tow, lamp, linedrawing

Description automatically generated

**Lesson 1**

C4: Real vector spaces

* These are defined by axioms (assumptions)
* There are different types of vector spaces: Real, Polynomial, subspaces,

**Operations on Real vector spaces**

|  |  |
| --- | --- |
| Addition | Given two elements x, y in X, one can form the sum x+y, which is also an element of X. |
| Inverse | Given an element x in X, one can form the inverse -x, which is also an element of X. |
| Scalar multiplication | Given an element x in X and a real number c, one can form the product cx, which is also an element of X |

Axioms on Vector Spaces

* Ten axioms (assumptions)
* 8 of these are properties of vectors in

**Additive Axioms**

1.

*If and are objects in ,*

*Then is in*

2.

3.

4. *There is an object in , called a zero vector for*

*such that*

5. *For each in*

*There is an object in , called a negative of*

*such that*

**Distributive Axioms**

6.

*If is any scalar and is any object in*

*then is in*

7.

8.

**Multiplicative Axioms**

9.

10.

Example: Is a vector space?

Yes.

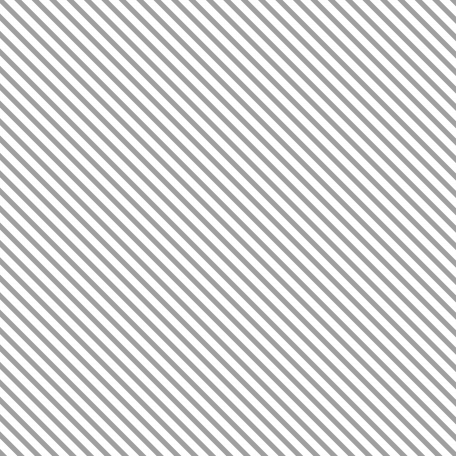
Example: Is a vector space?

No. Via Axiom 6 the vector would be outside

Chart, line chart

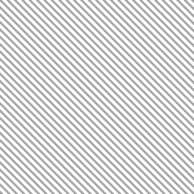
Description automatically generatedChart, table

Description automatically generated



**✘**

**✓**



6.

*If is any scalar and is any object in then is in*

e.g. let

1.

*If and are objects in ,*

*Then is in*

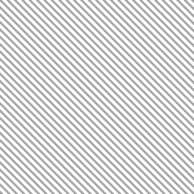
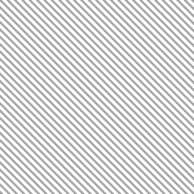
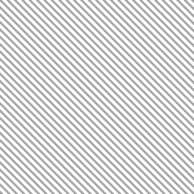
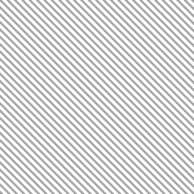
Example: Is a vector space?

No. Via Axiom 6 the vector would be outside

Chart, line chart

Description automatically generatedChart, line chart

Description automatically generated



**✘**

**✓**

6.

*If is any scalar and is any object in then is in*

1.

*If and are objects in ,*

*Then is in*

**Lesson 2**

C4: Real vector spaces

Polynomials

is a vector space

* The degree of a polynomial is the highest power with a non-zero coefficient

*Assuming*

* The polynomial with degree is the zero polynomial

Axioms on Polynomial Vector Spaces

*Assuming*

1.

*Two vector spaces are in independently.*

*Their sum is also in*

e.g.

Rewritten:

2.

3. for all

**Lesson 3**

C4: Real vector spaces

Subspaces

- Subspaces are also vector spaces

*is a subspace of*

A subspace of a vector is a subset that satisfies three conditions:

1. The zero vector is in

2. is closed under addition

*A set is closed under addition if you can add any two numbers in the set and still have a number in the set as a result*

then

3. is closed under multiplication of scalars.

defined on

then

Some notable examples:

A subspace is also a vector space **✓**

A vector space is also a subspace **✓**

is a subspace of ✘

✘ Not a

Subspace!

Remember every vector in a subspace must come from the vector space

**✓** is a

Subspace!

<https://www.youtube.com/watch?v=TuOQ4-kmcE0>

Example: Is a subspace of

Yes.

1. **✓**
2. **✓**
3. **✓**

Example: Let .

Show that is a subspace of

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

1. **✓**
2. Define and

**✓**

*This is the span*

1. **✓**

Remember that we defined

*Also in the span*

Example: Let be the set of all vectors of the form

Show that is a subspace or find a counterexample.

Reduce this to a span of something

**✓**

**Lesson 4**

C4: Real vector spaces

Span of vectors

* Kind of like the cross product of

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

With weights

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors

With weights

Example:

Let

*Write down all the possible combinations*

Chart, line chart

Description automatically generated

Background pattern

Description automatically generated Unit vectors.

So the x axis is the plane

And the y axis is the plane

*Therefore, the span of our vectors and is the whole plane*

AKA: *the vectors and span the whole of*

Example:

List 3 vectors in where and

OR

OR

OR

OR

etc.

*Just remember to list ALL possible* linear combinations

Example:

Given

*matrix vector*

Is in

*B would be a linear combination of and*

1. Make an augmented matrix with and

1. Solve augmented matrix

*Add 2x row one to row 3*

*Add 2x row one to row 3, multiply row by -1*

Solutions

R3:

R2:

R1:

**Lesson 5**

C4: Real vector spaces

Linear combinations

**Span**

If are in

then the set of all linear combinations of

is denoted by

is the collection of ALL vectors of the form

With weights

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors

With weights

Example:

is a linear combination

AKA

Some notable examples:

Another notation for vector is ✘

*The vector*  *is 1 dimensional*

*The vector 2 dimensional (2 vectors)*

A linear combination of and is **✓**

*Remember that a linear combination is just a scalar a vector*

*Would be the same as*

is a vector in ✘

*Corresponds to the points so should be represented in a 3-dimensional space*

Example

Let

*is just a linear combination of and*

*is also a linear combination of and . We would probably assign a value to represent . Therefore, let*

Chart, line chart

Description automatically generatedBackground pattern

Description automatically generated

Example

Let

Can be generated as a linear combination of and ?

1. Make an augmented matrix with and ?

**Wolframalpha**

row reduce [(-1,-2,-7), (2,-5,-4), (-5,-6,-3)]

1. Solve for augmented matrix

Solutions:

R2:

R1:

1. Substitute solutions into and

**✓**

A vector equation

As the same solution set as the linear system whose augmented matrix is

[]

**Lesson 6**

C4: Real vector spaces

Linear independence

<https://www.youtube.com/watch?v=XI2kYIxhe-o>

A set of vectors in is linearly independent if

Has the only trivial solution

*The homogenous system (i.e., of the form ) has only the trivial solution*

If a non-trivial solution exists, then the set of vectors is linearly dependent

*If there is a free variable, then the vectors are linearly independent*

We don’t want any redundant vectors!

*Main diagonal must be nonzero entries*

e.g.

Some notable examples:

If contains , then the set is linearly independent

A set of two vectors is linearly independent when

*Vector v1 is NOT equal to some multiple of v2*

**✘**

Example:

Let

**✓**

Example:

Let

is linearly dependent if

*If a set has more vectors than there are entries in each vector, then the set is linearly dependent*.

Example:

**✘**

Let

2 dimensions, 3 vectors.

Linearly dependent

Example:

Let

Chart, line chart

Description automatically generated

**✘**

The set is linearly dependent

*We can get to the point using and*

*is redundant, it to get to any new points*

Example:

Let

Chart, line chart

Description automatically generated

**✓**

The set is linearly independent

Example:

Are the following sets linearly independent?

Let

1. Make an augmented matrix with , and

1. Solve for augmented matrix

swap R1 and R2

Solutions:

R3:

R2:

R1:

Trivial solution

No free variable. Therefore, linearly independent

**Lesson 7**

C4: Real vector spaces

Basis and Dimension

<https://www.youtube.com/watch?v=OLqc_rt7abI>

Let be a subspace of vector space .

is a basis for if:

1. is linearly independent

*Linear combinations*

Some notable examples:

is the standard basis for

unit vectors

1. all linearly independent
2. all in

is the standard basis for (polynomials)

1. all linearly independent
2. all in

Example:

Determine if is a basis for

1. Make an augmented matrix with the set of vectors

1. Solve for augmented matrix

Using the above, Let

**✘**

Not linearly independent.

Because

Example:

Find a basis for the set of vectors in on the line

Rewrite the above as

1. all linearly independent

Dimension

[https://www.youtube.com/watch?v=zr\_iMt0k5TQ](https://www.youtube.com/watch?v=zr_iMt0k5TQ&list=PLDDGPdw7e6AjJacaEe9awozSaOou-NIx_&index=39)

If is spanned by a finite set,

then is finite dimensional

The dimension of is the number of vectors in a basis for

*TDLR:*

*is the vector*

*is the subspace in*

*is the basis of*

*is the number of vectors in*

Some notable examples:

*remember standard basis*

*remember standard basis*

Let be a subspace of the finite dimensional vector space .

Any linearly independent set in can be expanded to a basis for

and

**of nullspace and column space**

The dimension of is the number of free variables in

The dimension of is the number of pivot columns in

*Pivot columns: columns that contain the leading 1’s of the rows*

e.g., 1,2 and 4 are the pivot columns

Example:

2x4 matrix

2 pivots

2 free variables

Example:

Find the dimensions of the subspace spanned by

* Instinctively each vector has 3 points, so they’re probably in
* Because of this, it is 3-dimensional so
* Need to still prove because it might actually be subspace in a plane in

1. Make an augmented matrix with the set of vectors

1. Solve for augmented matrix

…

We see that

So, is the redundant vector.

From first attempt

Find the basis for the row space of (i.e. )

Also one of the subspaces of

**Lesson 8**

C4: Real vector spaces

Change of basis

<https://www.youtube.com/watch?v=VCZetCt7vA0>

*Not the same as basis of a matrix*

Let and be bases of vector space .

**Basis**

Let be a subspace of vector space .

is a basis for if:

1. is linearly independent

*Linear combinations*

is the standard basis for

Then

Some vector relative to

Some vector relative to

For some matrix and

*Change of basis matrix is an matrix that takes a vector relative to and maps it to a vector relative to*

This looks familiar to our standard basis

Example:

Let

Where and

Find

*Find the change of basis matrix*

Row reduction

1. Use a partition matrix so that

Vectors of C are on the left side

Vectors of B are on the right side

Reduce C to the identity

1. Write in reduced echelon form (RREF)

treat as a normal matrix

Example: Polynomials

Given

Remember that is the standard basis for (polynomials)

Find

1. Write in reduced echelon form (RREF)

is

is

is

**Lesson 9**

C4: Real vector spaces

Row space, column space and Null Space

<https://www.youtube.com/watch?v=JlC58uaJVsg>

**Nullspace**

The Null Space of an matrix , is the set of all solutions to



*Set of solutions in that map to*

*Null space = no of non-pivot columns*

Some notable examples:

If is , is a subspace of

Example:

Is in where

Using the definition, we just need to check that

times first column

times second column

times third column

**✓**

Finding explicit solutions to

Example:

Find the spanning set of

1. Write in reduced echelon form (RREF)

**Wolframalpha**

rref({1,3,5,0},{0,1,4,-2})

1. Compute general solution

is free

is free

1. Vector form solution

1. Find the basis for the row space of A

**Column space**

If is , and

*Column space is basically a span*

*Remember that a span is a subspace*

Some notable examples:

If is , is a subspace of

What happens if has a solution for every in

No matter you use, you will always get

*Like a linear combination*

The span will be all of

Example:

Find a matrix such that

*We need to find a matrix a that produces the span*

Separate the vector in terms of

and

*Like a linear combination*

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors With weights

Let

The span of the above is a linear combination of the first and second column

Example:

Find so that

If is a subspace of , what is ?

*We need to find a matrix a that produces the span*

*What is the dimension that is a subspace of?*

Let

matrix

*Should be a subspace of*

**Row Space**

<https://www.youtube.com/watch?v=Q7x5So3suYI>

The set of all linear combinations of row vectors is called the row space

Row is a subspace of

We could write as

**wolframalpha**

transpose[(1,2,3,4), (0,1,2,3), (0,0,1,2)]

Example:

If matrices and are row equivalent, then

Find given

*A is equivalent to B*

*Different notation*

Rank

<https://www.youtube.com/watch?v=Q7x5So3suYI>

Some notable examples

**Rank Theorem**

Given ,

Example:

If and

What is

Example:

If the null space of a matrix is 5-dimensional, what is the rank of ?

Using Rank Theorem

Example:

If A is , what is the smallest possible dimension of ?

*Null space = no of non-pivot columns*

We could have 4 pivot columns i.e.,

**Nullity**

<https://www.youtube.com/watch?v=RCW13oqgUwk>

**The rank is the dimension of**

*No of Rows – non-pivot columns*

**The nullity is the dimension of**

*Set of solutions in that map to*

*Null space = no of non-pivot columns*

Some notable examples:

Given matrix

Example:

If a matrix has a rank of , what is the nullity?

If a matrix has a rank of , what is the nullity?

If a matrix has a nullity of , what is the rank?

If a matrix has a nullity of , what is the rank?

If is a matrix, what is the largest possible nullity of ?

The largest possible nullity is

Example:

Find the rank and nullity of the matrix

Steps:

1. RREF A

1. Find Rank

*Transpose matrix columns*

There are 3 nonzero rows in the row echelon form of the matrix

1. Use Rank-Nullity theorem

*We have a matrix*

**Lesson 11**

C5: Eigenvalues, Eigenvectors

<https://www.youtube.com/watch?v=ihscX_B1BoA>

**Determinants**

For any matrix the determinant is

ad

bc

Some notable examples:

Example:

**wolframalpha**

det({3,5},{1,2})

Find

**wolframalpha**

det({1,3,5},{2,1,1},{3,4,2})

*use wolfram to check final answer, might need to do workings yourself*

*If necessary, select the “row operations” option when calculating determinants*

Example:

…

Eigenvalues & Eigenvectors

1. An eigenvector of an matrix is a vector such that

For some scalar and

*works just like the scalar*

1. is an eigenvalue of iff has a non-trivial solution

i.e.,

*No linear independence. Will have a free variable in row reduced matrix*

*OR*

*For characteristic equations (basically the same thing)*

is an eigenvalue of iff

Some notable examples:

* The eigenvalues of a triangular matrix are the entries on the diagonal

*Applies to a matrix*

Example:

Find the eigenvalue of

*We want to find the lambda*

Factor out “2”

Example:

Suppose

Show that 2 is an eigenvalue of

*remember we want a non-trivial solution*

*the identity matrix*

1. Write in reduced echelon form (RREF)

…

*Free variable. Has a non-trivial solution*

2 is the eigenvalue of

But we could also find all the eigenvectors of

Try solve for

*Set of non-trivial solutions. It is the corresponding eigenvector*

**Characteristic equation & Eigenvalues**

*Determinants & eigenvalues are related!*

The characteristic equation of is the equation

Example:

Find the characteristic equation of

**wolframalpha**

({7,-2},{2,3})

**Characteristic Polynomials**

Proof:

If , and are similar, then they have the same characteristic polynomial

is similar to if and

for some invertible matrix

*We can do a transformation on and another transformation and get back to*

1. Prove that

*This should give us eigenvalues, and more importantly our characteristic polynomials*

assuming and are similar

**Remember**

**Remember**

**Eigenspaces**

<https://www.youtube.com/watch?v=8QHPlik7HAU>

Eigenvalues & Eigenvectors

1. An eigenvector of an matrix is a vector such that

For some scalar and

*works just like the scalar*

1. is an eigenvalue iff has a non-trivial solution

i.e.,

*No linear independence. Will have a free variable in row reduced matrix*

The set of all solutions is a subspace of called the eigenspace

i.e., or rather

NOTE: we find an eigenspace corresponding to an eigenvalue

Some notable examples:

If are eigenvectors corresponding to distinct eigenvalues

Then is linearly independent

**Linear Independence**

A set of vectors in is linearly independent if

Has the only trivial solution

*The homogenous system (i.e. of the form ) has only the trivial solution*

If a non-trivial solution exists, then the set of vectors is linearly dependent

*If there is a free variable, then the vectors are linearly independent*

We don’t want any redundant vectors!

Example:

Find a basis for the eigenspace of

corresponding to eigenvalue

*We need to find the solution to*

1. Write in reduced echelon form (RREF)

Solve

Diagram

Description automatically generated*Refer to example on Null spaces if you forget how to do this*

**Lesson 12**

C5: Eigenvalues, Eigenvectors Diagonalization

**Diagonalization**

We rewrite as where is the diagonal matrix

*Diagonal matrix: nonzero entries in the main diagonal*

, then

, then

*Just raise the diagonal entries to the power of*

Some notable examples:

* An matrix is diagonalizable

iff has linearly independent eigenvectors

Proof:

P: matrix, with linearly vectors

D: diagonal matrix, with nonzero diagonal entries

We can see that corresponds to and to

Therefore, is diagonalizable.

Theorem: DM

Theorem: DS

Theorem: DD

<https://www.youtube.com/watch?v=Xcln3xG8QGQ>

**The algebraic multiplicity of an Eigenvalue**

The number of times appears as a root of the characteristic polynomial

e.g., Given the characteristic equation

- Solutions: OR OR

- appears ONE times. It has a algebraic multiplicity of 1

- appears TWO times. It has a algebraic multiplicity of 2

**The geometric multiplicity of an Eigenvalue**

Let be an matrix

Let be one of the eigenvalues of

Then:

geometric multiplicity of algebraic multiplicity of

OR

*Dimension of the eigenspace for the eigenvalue*

*Remember:*

Example: Induction proof

Given

Show

1. Show that
2. Assume
3. Show

*Via induction hypothesis*

Via associativity

Remove identity

Example: NOV2016 Q3.2

In the exam

[1] Nullity

[2] Characteristic equation

[3] Basis for eigenspaces

[4] Diagonalization

Consider the matrix

[1] Rank:

Nullity:

[2] Show that the characteristic equation of is given by

*Using the top row for a, b and c*

*Multiply all expressions by*

[3] Find a basis for the eigenspace corresponding to each eigenvalue

*Find a null space of*

Eigenvalues

*Solve Characteristic equation*

OR OR

3.1 Solve For eigenspace corresponding to

*For the matrix , substitute eigenvalue into*

3.2 Find (set of solutions to )

* + 1. Write in reduced echelon form (RREF)

wolframalpha

rref ({0,1,1},{0,1,0},{0,1,1})

* + 1. Compute general solution

is free

*You can choose ANY value for in the vector*

*Chose 1 for this*

*Just write it like this*

3.1 Solve For eigenspace corresponding to

*For the matrix , substitute eigenvalue into*

3.2 Find (set of solutions to )

* + 1. Write in reduced echelon form (RREF)

wolframalpha

rref ({-1,1,1},{0,0,0},{0,1,0})

* + 1. Compute general solution

*You can choose ANY value for and in the vector*

*Chose 1 for this*

1. For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable?

**The algebraic multiplicity of an Eigenvalue**

The number of times appears as a root of the characteristic polynomial

**The geometric multiplicity of an Eigenvalue**

*Dimension of the eigenspace for the eigenvalue*

*Remember:*

Eigenvalues

OR OR

*appears TWO times*

The algebraic multiplicity of is

and the geometric multiplicity is

The algebraic multiplicity of is

*It appears TWO times as a root in our characteristic equation*

and the geometric multiplicity is

Thus is not diagonalizable (Theorem DM)

**Lesson 13**

C6:

Inner Products

*Same thing as dot product*

In the exam

[1] Inner Products

[2] Orthogonal with respect to the inner product

[3] Gram-Schmidt to find an orthogonal basis with respect to the inner product

**Dot product (***Euclidean Inner product)*

**General inner product**

Some properties:



1. Order doesn’t matter
2. Distributive
3. and

then

*V must be the zero vector i.e., this rule only applies to the zero vector*

A vector space with an inner product is an inner product space

**Inner products generated by matrices**

Let be vectors in

And let be an invertable matrix, then

Is the inner product on generated by

RHS

LHS

**Orthogonal**

If , then is orthogonal (perpendicular)

**Normalized**

**Orthonormal**

Both conditions above are satisfied (Orthogonal & Normalized)

Example:

Show that

Is an inner product on

*is a vector space*

Given

- For all

- For all and

Property 3

3.

- For all

**Lesson 14**

C6:

Angle and orthogonality in inner product spaces

**Lesson 15**

C6:

Orthonormal bases, Gram-Schmidt Process

**Lesson 16**

C6:

Best approximation, least squares

**Lesson 17**

C7: Diagonalization

orthogonal matrices

We already seen inverse matrices

We already seen transposed matrices

Orthogonal matrices make use of transposing

The purpose of Orthogonal matrices is to preserve lengths

Some Properties of Orthogonal Matrices:



*The inverse of an orthogonal matrix is equal to the transpose of the matrix*

* Rows, then columns

*An orthogonal matrix multiplied by its transpose is equal to the identity matrix*

* Columns, then rows

*A transpose of orthogonal matrix multiplied by its orthogonal matrix is equal to the identity matrix*

*Same as . The “norm” of*

*Remember that 𝑄.𝑄𝑇*

*remember,*

*Same as . The “norm” of*

* Norm

**An Orthogonal matrix is special because**

* its rows are orthonormal vectors
* its columns are orthonormal vectors

If the rows are different, we get 0 ORTHOGONAL

If the rows are the same, we get 1 NORMALIZED

If both of the above apply, the vector is ORTHONORMAL

**Lesson 18**

C7: Diagonalization

Orthogonal diagonalization

**Lesson 19**

C8: Linear Transformations

General Linear Transformations

Linear transformation: multiplying a vector by a matrix

It’s a function!

that maps

Same as

Has rows

Has rows

The range of is the set of all linear combinations of the columns of A

**Linear Combination**

Given are in and,

Scalars

The vector defined by

is called a linear combination of vectors With weights

Some notable examples:

When is a function linear?

Consequently

Example:

where

Function inputs a vector in and outputs a vector in

Find

*Same thing as*

Find an whose image under is

*should produce .*

*Essentially, we want to create the function*

1. Write as augmented matrix

1. Solve

…

R2:

R1:

produces

Example:

Suppose where

What does do to any vector geometrically?

Chart, line chart

Description automatically generated

ANSWER:

Reflection in the y-axis

**Lesson 20**

C8: Linear Transformations

Kernel and Range

**Lesson 21**

C8: Linear Transformations

Inverse linear transformations

**Lesson 22**

C8: Linear Transformations

Matrices of general linear transformations

**Lesson 23**

C8: Linear Transformations

Similarity